THE FIXED POINTS OF THE CIRCLE ACTION ON HOCHSCHILD HOMOLOGY

MARC HOYOIS

Let k be a commutative ring and let A be a flat k-algebra. The Hochschild complex $\mathrm{HH}(A)$ of A with coefficients in itself is defined as the normalization of a simplicial k-module $A^\natural$ with

$$A^\natural_n = A^{\otimes_k (n+1)}.$$

The simplicial k-module $A^\natural$ is in fact a *cyclic* k-module: it extends to a contravariant functor on Connes' cyclic category Λ. As we will see below, it follows that the chain complex $\mathrm{HH}(A)$ acquires an action of the circle group $\mathbb{T}$. The cyclic homology and negative cyclic homology of A over k are classically defined by means of explicit bicomplexes. The goal of this note is to show that:

(1) The *cyclic homology* $\mathrm{HC}(A)$ coincides with the *orbits* of the $\mathbb{T}$-action on $\mathrm{HH}(A)$.
(2) The *negative cyclic homology* $\mathrm{HN}(A)$ coincides with the *fixed points* of the $\mathbb{T}$-action on $\mathrm{HH}(A)$.

The first statement is due to Kassel [Kas87, Proposition A.5]. The second statement is probably well-known, but a proof seems to be missing from the literature. This gap was partially filled in [TV11], where (2) is proved at the level of connected components for A a smooth commutative k-algebras and $\mathbb{Q} \subset k$. Here we will give a proof of (1) and observe that (2) follows formally from a mild refinement of (1) and Koszul duality. This "mild refinement" amounts to identifying two classes in the second cohomology group of $\mathbb{CP}^\infty$.

We will proceed as follows:

- In §1, we recall abstract definitions of cyclic and negative cyclic homology in a more general context, namely for ∞-categories enriched in a symmetric monoidal ∞-category.
- In §2, we show that in the special case of differential graded categories over a commutative ring, the abstract definitions recover the classical ones.

1. Cyclic homology of enriched ∞-categories

Let $\mathcal{E}$ be a presentably *symmetric* monoidal ∞-category, for instance the ∞-category Mod_k for k an E_∞-ring. We denote by $\mathcal{C}\mathrm{at}(\mathcal{E})$ the ∞-category of $\mathcal{E}$-enriched ∞-categories with a set of objects. We will associate to every $\mathcal{C} \in \mathcal{C}\mathrm{at}(\mathcal{E})$ a cyclic object $\mathcal{C}^\natural$ in $\mathcal{E}$.

Given a finite directed graph $\mathcal{I}$, with vertices $\mathcal{I}_0$ and edges $\mathcal{I}_1$, define

$$\mathrm{Dia}(\mathcal{I}, \mathcal{C}) = \coprod_{f\colon \mathcal{I}_0 \to \mathrm{ob}(\mathcal{C})} \bigotimes_{e \in \mathcal{I}_1} \mathcal{C}(fe_0, fe_1) \in \mathcal{E}.$$

We think of $\mathrm{Dia}(\mathcal{I}, \mathcal{C})$ as an "object of $\mathcal{I}$-diagrams in $\mathcal{C}$". Its functoriality in $\mathcal{I}$ is described by the following category Υ:

- An object of Υ is a finite directed graph.
- A morphism $\mathcal{I} \to \mathcal{J}$ in Υ is a morphism of directed graphs from $\mathcal{I}$ to the free category on $\mathcal{J}$, such that every edge in $\mathcal{J}$ is passed through exactly once.

Given a morphism $\mathcal{I} \to \mathcal{J}$ in Υ, we can use composition in $\mathcal{C}$ to define a "restriction" map $\mathrm{Dia}(\mathcal{J}, \mathcal{C}) \to \mathrm{Dia}(\mathcal{I}, \mathcal{C})$, and we can upgrade this construction to a functor

$$\mathrm{Dia}\colon \Upsilon^{\mathrm{op}} \times \mathcal{C}\mathrm{at}(\mathcal{E}) \to \mathcal{E}.$$

A *cyclic graph* is a directed graph which is homeomorphic to an oriented circle. The cyclic category Λ [Con83] is equivalent to the full subcategory of Υ on the cyclic graphs (see Figure 1).

The cyclic object $\mathcal{C}^\natural$ is by definition the restriction of $\mathrm{Dia}(-, \mathcal{C})\colon \Upsilon^{\mathrm{op}} \to \mathcal{E}$ to cyclic graphs.

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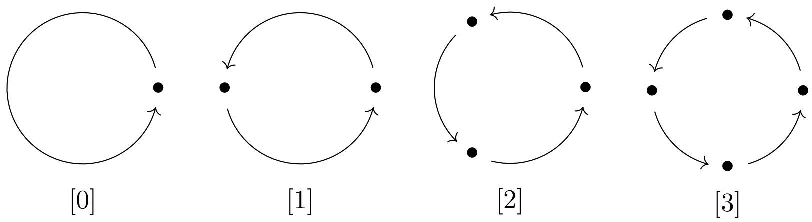

FIGURE 1. The cyclic category $\Lambda \subset \Upsilon$.

Remark 1.1. We can define an extension of Dia to graphs in which edges are labeled by bimodules. Given $\mathcal{C} \in \mathcal{C}\mathrm{at}(\mathcal{E})$ and a $\mathcal{C}$-bimodule $\mathcal{M}$, the cyclic graphs with one edge labeled by $\mathcal{M}$ and all other edges labeled by $\mathcal{C}$ form a subcategory equivalent to Δ. The restriction of Dia to this subcategory is the usual simplicial object in $\mathcal{E}$ whose colimit is the Hochschild homology of $\mathcal{C}$ with coefficients in $\mathcal{M}$.

Let $\Lambda \to \hat{\Lambda}$ denote the ∞-groupoid completion of Λ, and let $\mathbb{T}$ be the automorphism ∞-group of $[0]$ in $\hat{\Lambda}$. Since Λ is connected, there is a canonical equivalence $B\mathbb{T} \simeq \hat{\Lambda}$.

Let $\mathrm{PSh}(\Lambda, \mathcal{E})$ be the ∞-category of $\mathcal{E}$-valued presheaves on Λ, and let $\mathrm{PSh}_{\simeq}(\Lambda, \mathcal{E}) \subset \mathrm{PSh}(\Lambda, \mathcal{E})$ be the full subcategory of presheaves sending all morphisms of Λ to equivalences. We have an obvious equivalence

$$\mathrm{PSh}_{\simeq}(\Lambda, \mathcal{E}) \simeq \mathrm{PSh}(B\mathbb{T}, \mathcal{E}).$$

Since $\mathcal{E}$ is presentable and Λ is small, $\mathrm{PSh}_{\simeq}(\Lambda, \mathcal{E})$ is a reflective subcategory of $\mathrm{PSh}(\Lambda, \mathcal{E})$. We denote by

$$|-|\colon \mathrm{PSh}(\Lambda, \mathcal{E}) \to \mathrm{PSh}_{\simeq}(\Lambda, \mathcal{E}) \simeq \mathrm{PSh}(B\mathbb{T}, \mathcal{E})$$

the left adjoint to the inclusion. The morphisms

$$* \xrightarrow{i} B\mathbb{T} \xrightarrow{p} *$$

each induce three functors between the categories of presheaves. We will write

$$u_{\mathbb{T}} = i^*, \quad (-)_{h\mathbb{T}} = p_!, \quad (-)^{h\mathbb{T}} = p_*$$

for the forgetful functor, the $\mathbb{T}$-orbit functor, and the $\mathbb{T}$-fixed points functor, respectively.

Lemma 1.2. *Let $F \in \mathrm{PSh}(\Lambda, \mathcal{E})$ be a cyclic object. There is a natural equivalence*

$$u_{\mathbb{T}}|F| \simeq \operatorname*{colim}_{[n] \in \Delta^{\mathrm{op}}} F([n]).$$

Proof. Let $j\colon \Delta \hookrightarrow \Lambda$ be the inclusion. Let $F \in \mathrm{PSh}_{\simeq}(\Delta, \mathcal{E})$ and let $j_*F \in \mathrm{PSh}(\Lambda, \mathcal{E})$ be the right Kan extension of F. Since every morphism in Λ is a composition of isomorphisms and morphisms in Δ, j_*F sends all morphisms in Λ to equivalences. Thus, we have a commuting square

$$\begin{tikzcd} \mathrm{PSh}_{\simeq}(\Delta, \mathcal{E}) \arrow[r, hook] \arrow[d, "j_*"'] & \mathrm{PSh}(\Delta, \mathcal{E}) \arrow[d, "j_*"] \\ \mathrm{PSh}_{\simeq}(\Lambda, \mathcal{E}) \arrow[r, hook] & \mathrm{PSh}(\Lambda, \mathcal{E}). \end{tikzcd}$$

Since Δ^{op} is sifted, evaluation at $[0]$ is an equivalence $\mathrm{PSh}_{\simeq}(\Delta, \mathcal{E}) \simeq \mathcal{E}$. The left adjoint square, followed by evaluation at $[0]$, says that $u_{\mathbb{T}}|F| \simeq \operatorname{colim} j^*F$, as desired. $\square$

It follows from Lemma 1.2 that $u_{\mathbb{T}}|\mathcal{C}^\natural| \simeq \mathrm{HH}(\mathcal{C})$, the Hochschild homology of $\mathcal{C}$ with coefficients in itself. As another corollary, we recover the following computation of Connes [Con83, Théorème 10]:

Corollary 1.3. $\hat{\Lambda} \simeq K(\mathbb{Z}, 2)$.

Proof. If $F \in \mathrm{PSh}_{\simeq}(\Lambda)$, then, by Yoneda, $\mathrm{Map}(\Lambda^0, F) \simeq \mathrm{Map}(\hat{\Lambda}^0, F)$. In other words, the canonical map $\Lambda^0 \to \hat{\Lambda}^0$ induces an equivalence $|\Lambda^0| \simeq \hat{\Lambda}^0$, and hence $u_{\mathbb{T}}|\Lambda^0| \simeq \mathbb{T}$. On the other hand, the underlying simplicial set of Λ^0 is $\Delta^1/\partial\Delta^1$, so $u_{\mathbb{T}}|\Lambda^0| \simeq K(\mathbb{Z}, 1)$ by Lemma 1.2. Thus, $\mathbb{T}$ is a $K(\mathbb{Z}, 1)$, and hence $B\mathbb{T} \simeq \hat{\Lambda}$ is a $K(\mathbb{Z}, 2)$. $\square$

In particular, $\mathbb{T}$ is equivalent to the circle as an ∞-group, which justifies the notation.

Definition 1.4. Let $\mathcal{E}$ be a presentably symmetric monoidal ∞-category and let $\mathcal{C} \in \mathcal{C}\mathrm{at}(\mathcal{E})$.

(1) The *cyclic homology* of $\mathcal{C}$ is

$$\mathrm{HC}(\mathcal{C}) = |\mathcal{C}^\natural|_{h\mathbb{T}} \in \mathcal{E}.$$

(2) The *negative cyclic homology* of $\mathcal{C}$ is

$$\mathrm{HN}(\mathcal{C}) = |\mathcal{C}^\natural|^{h\mathbb{T}} \in \mathcal{E}.$$

Note that $\mathrm{HC}(\mathcal{C})$ is simply the colimit of $\mathcal{C}^\natural\colon \Lambda^{\mathrm{op}} \to \mathcal{E}$.

Remark 1.5. There are several interesting refinements and generalizations of the above definitions. Note that the invariants HH, HC, and HN depend only on $|\mathcal{C}^\natural|$. The *topological cyclic homology* of $\mathcal{C}$ is a refinement of negative cyclic homology, defined when $\mathcal{E}$ is the ∞-category of modules over an E_∞-ring, which uses some additional structure on $\mathcal{C}^\natural$. In another direction, additional structure on $\mathcal{C}$ can lead to $|\mathcal{C}^\natural|$ being acted on by more complicated ∞-groups. For example, if $\mathcal{C}$ has a duality $\dagger$, then $\mathcal{C}^\natural$ extends to the dihedral category whose classifying space is $B\mathrm{O}(2)$. The coinvariants $|\mathcal{C}^\natural|_{h\mathrm{O}(2)}$ are called the *dihedral homology* of $(\mathcal{C}, \dagger)$.

The previous definitions apply in particular when $\mathcal{C}$ has a unique object, in which case we may identify it with an A_∞-algebra in $\mathcal{E}$. If A is an E_∞-algebra in $\mathcal{E}$, there is a more direct description of $|A^\natural|$. In this case, $A^\natural$ is the underlying cyclic object of the cyclic E_∞-algebra $\Lambda^0 \otimes A \in \mathrm{PSh}(\Lambda, \mathrm{CAlg}(\mathcal{E}))$, where Λ^0 is the cyclic set represented by $[0] \in \Lambda$ and $\otimes$ is the canonical action of the ∞-category $\mathcal{S}$ of spaces on the presentable ∞-category $\mathrm{CAlg}(\mathcal{E})$. For any cyclic space $K \in \mathrm{PSh}(\Lambda)$, we clearly have $|K \otimes A| \simeq |K| \otimes A$. It follows that $|A^\natural| \in \mathrm{PSh}(B\mathbb{T}, \mathcal{E})$ is the underlying object of the E_∞-algebra

$$|\Lambda^0| \otimes A \simeq \mathbb{T} \otimes A \in \mathrm{PSh}(B\mathbb{T}, \mathrm{CAlg}(\mathcal{E})).$$

In particular, $\mathrm{HH}(A)$ and $\mathrm{HN}(A)$ inherit E_∞-algebra structures from A. Their geometric interpretation is the following: if $X = \operatorname{Spec} A$, then $\operatorname{Spec}\mathrm{HH}(A)$ is the *free loop space* of X and $\operatorname{Spec}\mathrm{HN}(A)$ is the *space of circles* in X. The cyclic homology $\mathrm{HC}(A)$ is a quasi-coherent sheaf on the free loop space of X.

2. Comparison with the classical definitions

Let k be a discrete commutative ring and let A be an A_∞-algebra over k. The cyclic and negative cyclic homology of A over k are classically defined via explicit bicomplexes. Let us start by recalling these definitions, following [Lod92, §5.1].

Let $M_\bullet$ be a cyclic object in an additive category $\mathcal{A}$. The usual presentation of Λ provides the face and degeneracy operators $d_i\colon M_n \to M_{n-1}$ and $s_i\colon M_n \to M_{n+1}$ $(0 \le i \le n)$, as well as the cyclic operator $c\colon M_n \to M_n$ of order $n+1$. We define the additional operators

$$\begin{aligned}
b\colon M_n \to M_{n-1}, &\quad b = \sum_{i=0}^{n} (-1)^i d_i,\\
s_{-1}\colon M_n \to M_{n+1}, &\quad s_{-1} = c s_n,\\
t\colon M_n \to M_n, &\quad t = (-1)^n c,\\
N\colon M_n \to M_n, &\quad N = \sum_{i=0}^{n} t^i,\\
B\colon M_n \to M_{n+1}, &\quad B = (\mathrm{id} - t) s_{-1} N.
\end{aligned}$$

We easily verify that $b^2 = 0$, $B^2 = 0$, and $bB + Bb = 0$. In particular, (M, b) is a chain complex in $\mathcal{A}$. We now take $\mathcal{A}$ to be the category Ch_k of chain complexes of k-modules. Then (M, b) is a (commuting) bicomplex and we denote by $(C_*(M), b)$ the total chain complex with

$$C_n(M) = \bigoplus_{p+q=n} M_{p,q}.$$

We then form the (anticommuting) bicomplex

$$\begin{tikzcd}
& \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
\cdots & C_2(M) \arrow[l] \arrow[d, "b"'] & C_1(M) \arrow[l, "B"'] \arrow[d, "b"'] & C_0(M) \arrow[l, "B"'] \arrow[d, "b"'] & \cdots \arrow[l] \\
\cdots & C_1(M) \arrow[l] \arrow[d, "b"'] & C_0(M) \arrow[l, "B"'] \arrow[d, "b"'] & C_{-1}(M) \arrow[l, "B"'] \arrow[d, "b"'] & \cdots \arrow[l] \\
\cdots & C_0(M) \arrow[l] \arrow[d] & C_{-1}(M) \arrow[l, "B"'] \arrow[d] & C_{-2}(M) \arrow[l, "B"'] \arrow[d] & \cdots \arrow[l] \\
& \vdots & \vdots & \vdots &
\end{tikzcd}$$

where the $C_0(M)$'s are on the main diagonal. Removing all negatively graded columns, we obtain the *cyclic bicomplex* $\mathrm{BC}(M)$; removing all the positively graded columns, we obtain the *negative cyclic bicomplex* $\mathrm{BN}(M)$. Finally, we form the total complexes

$$\mathrm{Tot}^{\oplus}\,\mathrm{BC},\ \mathrm{Tot}^{\Pi}\,\mathrm{BN}\colon \mathrm{PSh}(\Lambda, \mathrm{Ch}_k) \to \mathrm{Ch}_k,$$

where

$$\mathrm{Tot}^{\oplus}(C)_n = \bigoplus_{p+q=n} C_{p,q} \quad \text{and} \quad \mathrm{Tot}^{\Pi}(C)_n = \prod_{p+q=n} C_{p,q}.$$

These functors clearly preserve quasi-isomorphisms and hence induce functors

$$\mathrm{CC},\ \mathrm{CN}\colon \mathrm{PSh}(\Lambda, \mathrm{Mod}_k) \to \mathrm{Mod}_k.$$

Theorem 2.1. *Let k be a discrete commutative ring and M a cyclic k-module. Then there are natural equivalences*

$$|M|_{h\mathbb{T}} \simeq \mathrm{CC}(M) \quad \textit{and} \quad |M|^{h\mathbb{T}} \simeq \mathrm{CN}(M).$$

In particular, if $\mathcal{C}$ is a k-linear ∞-category, then

$$\mathrm{HC}(\mathcal{C}) \simeq \mathrm{CC}(\mathcal{C}^\natural) \quad \textit{and} \quad \mathrm{HN}(\mathcal{C}) \simeq \mathrm{CN}(\mathcal{C}^\natural).$$

We first rephrase the classical definitions in terms of *mixed complexes*, following Kassel [Kas87]. We let $k[\epsilon]$ be the differential graded k-algebra

$$\cdots \to 0 \to k\epsilon \xrightarrow{0} k \to 0 \to \cdots,$$

which is nonzero in degrees 1 and 0. The ∞-category $\mathrm{Mod}_{k[\epsilon]}$ is the localization of the category of differential graded $k[\epsilon]$-modules, also called mixed complexes, at the quasi-isomorphisms. We denote by

$$K\colon \mathrm{PSh}(\Lambda, \mathrm{Mod}_k) \to \mathrm{Mod}_{k[\epsilon]}$$

the functor induced by sending a cyclic chain complex M to the mixed complex $(C_*(M), b, B)$.

Lemma 2.2. *Let $M \in \mathrm{PSh}(\Lambda, \mathrm{Mod}_k)$. Then*

$$\mathrm{CC}(M) \simeq k \otimes_{k[\epsilon]} K(M) \quad \textit{and} \quad \mathrm{CN}(M) \simeq \mathrm{Hom}_{k[\epsilon]}(k, K(M)).$$

Proof. We work at the level of complexes. Let Qk be the nonnegatively graded mixed complex

$$\cdots \leftrightarrows k\epsilon \underset{0}{\overset{\epsilon}{\leftrightarrows}} k \underset{\epsilon}{\overset{0}{\leftrightarrows}} k\epsilon \underset{0}{\overset{\epsilon}{\leftrightarrows}} k.$$

There is an obvious morphism $Qk \to k$ which is a cofibrant resolution of k for the projective model structure on mixed complexes. By inspection, we have isomorphisms of chain complexes

$$\mathrm{Tot}^{\oplus}\,\mathrm{BC}(M) \simeq Qk \otimes_{k[\epsilon]} (C_*(M), b, B) \quad \text{and} \quad \mathrm{Tot}^{\Pi}\,\mathrm{BN}(M) \simeq \mathrm{Hom}_{k[\epsilon]}(Qk, (C_*(M), b, B)).$$

This proves the claim. ∎

Let $k[\mathbb{T}]$ be the A_∞-ring $k \otimes \Sigma^\infty \mathbb{T}_+$. There is an obvious equivalence of ∞-categories

$$\mathrm{PSh}(B\mathbb{T}, \mathrm{Mod}_k) \simeq \mathrm{Mod}_{k[\mathbb{T}]}$$

that makes the following squares commute:

$$\begin{tikzcd} \mathrm{PSh}(B\mathbb{T}, \mathrm{Mod}_k) \arrow[r, "(-)_{h\mathbb{T}}"] \arrow[d, "\simeq"'] & \mathrm{Mod}_k \arrow[d, equal] \\ \mathrm{Mod}_{k[\mathbb{T}]} \arrow[r, "k \otimes_{k[\mathbb{T}]} -"] & \mathrm{Mod}_k, \end{tikzcd} \qquad \begin{tikzcd} \mathrm{PSh}(B\mathbb{T}, \mathrm{Mod}_k) \arrow[r, "(-)^{h\mathbb{T}}"] \arrow[d, "\simeq"'] & \mathrm{Mod}_k \arrow[d, equal] \\ \mathrm{Mod}_{k[\mathbb{T}]} \arrow[r, "\mathrm{Hom}_{k[\mathbb{T}]}(k, -)"] & \mathrm{Mod}_k. \end{tikzcd} \tag{2.3}$$

Let $\gamma \in H_1(\mathbb{T}, \mathbb{Z})$ be a generator. Sending ϵ to γ defines an equivalence of augmented A_∞-k-algebras $\gamma\colon k[\epsilon] \simeq k[\mathbb{T}]$, whence an equivalence of ∞-categories

$$\gamma^*\colon \mathrm{Mod}_{k[\mathbb{T}]} \simeq \mathrm{Mod}_{k[\epsilon]}.$$

The main result of this note is then that $K(M)$ is a model for $|M|$. More precisely:

Theorem 2.4. *There exists a generator $\gamma \in H_1(\mathbb{T}, \mathbb{Z})$ such that the following triangle commutes:*

$$\begin{tikzcd} \mathrm{PSh}(\Lambda, \mathrm{Mod}_k) \arrow[r, "|-|"] \arrow[dr, "K"'] & \mathrm{PSh}(B\mathbb{T}, \mathrm{Mod}_k) \arrow[d, "\simeq"', "\gamma^*"] \\ & \mathrm{Mod}_{k[\epsilon]}. \end{tikzcd}$$

Theorem 2.1 follows from Theorem 2.4, Lemma 2.2, and (2.3). Note that K sends $\mathrm{colim}_{\Delta^{\mathrm{op}}}$-equivalences to equivalences and hence factors through $|-|$. By Theorem 2.4, the induced functor $\mathrm{PSh}(B\mathbb{T}, \mathrm{Mod}_k) \to \mathrm{Mod}_{k[\epsilon]}$ agrees with γ^*. In particular, we recover the following result of Dwyer and Kan [DK85, Remark 6.7]:

Corollary 2.5. *The functor $K\colon \mathrm{PSh}(\Lambda, \mathrm{Mod}_k) \to \mathrm{Mod}_{k[\epsilon]}$ induces an equivalence of ∞-categories*

$$\mathrm{PSh}_{\simeq}(\Lambda, \mathrm{Mod}_k) \simeq \mathrm{Mod}_{k[\epsilon]}.$$

To prove Theorem 2.4, we consider the "universal case", namely the cocyclic cyclic k-module $k[\Lambda^\bullet]$. We have a natural equivalence

$$M \simeq k[\Lambda^\bullet] \otimes_\Lambda M,$$

where

$$\otimes_\Lambda\colon\ \mathrm{Fun}(\Lambda, \mathrm{Mod}_k) \times \mathrm{PSh}(\Lambda, \mathrm{Mod}_k) \to \mathrm{Mod}_k$$

is the coend pairing. Similarly, we have

$$|M| \simeq |k[\Lambda^\bullet]| \otimes_\Lambda M \quad \text{and} \quad K(M) \simeq K(k[\Lambda^\bullet]) \otimes_\Lambda M,$$

since both $|-|$ and K commute with tensoring with constant k-modules and with colimits (for K, note that colimits in $\mathrm{Mod}_{k[\epsilon]}$ are detected by the forgetful functor to Mod_k). Thus, it will suffice to produce an equivalence of cocyclic $k[\epsilon]$-modules

$$\gamma^*|k[\Lambda^\bullet]| \simeq K(k[\Lambda^\bullet]). \tag{2.6}$$

Let $k[u]$ denote the A_∞-k-coalgebra $k \otimes_{k[\epsilon]} k$. Note that a $k[u]$-comodule structure on $M \in \mathrm{Mod}_k$ is the same thing as map $M \to M[2]$. The functor $k \otimes_{k[\epsilon]} -\colon \mathrm{Mod}_{k[\epsilon]} \to \mathrm{Mod}_k$ factors through a fully faithful functor from $k[\epsilon]$-modules to $k[u]$-comodules:

$$\begin{tikzcd} & \mathrm{Comod}_{k[u]} \arrow[d, "\text{forget}"] \\ \mathrm{Mod}_{k[\epsilon]} \arrow[ur, hook, dashed] \arrow[r, "k \otimes_{k[\epsilon]} -"'] & \mathrm{Mod}_k. \end{tikzcd}$$

To prove (2.6), it will therefore suffice to produce an equivalence of cocyclic $k[u]$-comodules

$$k \otimes_{k[\epsilon]} \gamma^*|k[\Lambda^\bullet]| \simeq k \otimes_{k[\epsilon]} K(k[\Lambda^\bullet]). \tag{2.7}$$

Note that both cocyclic objects send all morphisms in Λ to equivalences and hence can be viewed as functors $B\mathbb{T} \to \mathrm{Comod}_{k[u]}$.

Let us first compute the left-hand side of (2.7). The generator γ induces an equivalence of coaugmented A_∞-k-coalgebras $\check{\gamma}\colon k[u] \simeq k[B\mathbb{T}]$, whence an equivalence of ∞-categories

$$\check{\gamma}^*\colon \mathrm{Comod}_{k[B\mathbb{T}]} \simeq \mathrm{Comod}_{k[u]}.$$

We clearly have

$$k \otimes_{k[\epsilon]} \gamma^*|k[\Lambda^\bullet]| \simeq \check{\gamma}^*|k[\Lambda^\bullet]|_{h\mathbb{T}}.$$

Now, $|k[\Lambda^\bullet]|_{h\mathbb{T}} \simeq k[|\Lambda^\bullet|_{h\mathbb{T}}]$, where $|\Lambda^\bullet|_{h\mathbb{T}}$ is a $B\mathbb{T}$-comodule in $\mathrm{Fun}_{\simeq}(\Lambda, \mathcal{S}) \simeq \mathcal{S}_{/B\mathbb{T}}$. If $\pi^*\colon \mathcal{S} \to \mathcal{S}_{/B\mathbb{T}}$ is the functor $\pi^*X = X \times B\mathbb{T}$, then a $B\mathbb{T}$-comodule structure on π^*X is simply a map $\pi^*X \to \pi^*B\mathbb{T}$, i.e., a map $X \times B\mathbb{T} \to B\mathbb{T}$ in $\mathcal{S}$. Here, $|\Lambda^\bullet|_{h\mathbb{T}}$ is $\pi^*(*) \in \mathcal{S}_{/B\mathbb{T}}$ and its $B\mathbb{T}$-comodule structure $\sigma\colon \pi^*(*) \to \pi^*(B\mathbb{T})$ is given by the identity $B\mathbb{T} \to B\mathbb{T}$. Applying $\check{\gamma}^*k[-]$, we deduce that the left-hand side of (2.7) is the constant cocyclic k-module $\underline{k}$ with $k[u]$-comodule structure given by the composition

$$\underline{k} \xrightarrow{\sigma} \underline{k}[B\mathbb{T}] \overset{\check{\gamma}}{\simeq} \underline{k}[u]. \tag{2.8}$$

Note that equivalence classes of $k[u]$-comodule structures on $\underline{k}$ are in bijection with

$$[\underline{k}, \underline{k}[2]] \simeq H^2(B\mathbb{T}, k).$$

Under this classification, (2.8) comes from an integral cohomology class, namely the image of the identity $B\mathbb{T} \to B\mathbb{T}$ under the isomorphism

$$[B\mathbb{T}, B\mathbb{T}] \overset{\check{\gamma}}{\simeq} H^2(B\mathbb{T}, \mathbb{Z}).$$

In particular, it comes from a generator of the infinite cyclic group $H^2(B\mathbb{T}, \mathbb{Z})$, determined by γ. We must therefore show that the right-hand side of (2.7) is also equivalent to the constant cocyclic k-module $\underline{k}$ with $k[u]$-comodule structure classified by a generator of $H^2(B\mathbb{T}, \mathbb{Z})$.

Recall that $K(k[\Lambda^\bullet])$ is the following mixed complex of cocyclic k-modules:

$$\cdots \rightleftarrows k[\Lambda_2] \underset{B}{\overset{b}{\rightleftarrows}} k[\Lambda_1] \underset{B}{\overset{b}{\rightleftarrows}} k[\Lambda_0].$$

Consider the mixed complex Qk from the proof of Lemma 2.2, which can be used to compute $k \otimes_{k[\epsilon]} -$ at the level of complexes. It comes with an obvious self-map $Qk \to Qk[2]$ which induces the $k[u]$-comodule structure on $k \otimes_{k[\epsilon]} M$ for every mixed complex M. Let us write down explicitly the resulting chain complex $Qk \otimes_{k[\epsilon]} K(k[\Lambda^\bullet])$ of cocyclic $k[u]$-comodules. It is the total complex of the first-quadrant bicomplex

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] \\
k[\Lambda_2] \arrow[d, "b"'] & k[\Lambda_1] \arrow[l, "B"'] \arrow[d, "b"'] & k[\Lambda_0] \arrow[l, "B"'] \\
k[\Lambda_1] \arrow[d, "b"'] & k[\Lambda_0] \arrow[l, "B"'] & \\
k[\Lambda_0], & &
\end{tikzcd} \tag{2.9}$$

with $k[u]$-comodule structure induced by the obvious degree $(-1,-1)$ endomorphism δ.

Lemma 2.10. *The bicomplex* (2.9) *is a resolution of the constant cocyclic k-module $\underline{k}$. Moreover, the endomorphism δ represents a generator of the invertible k-module $[\underline{k}, \underline{k}[2]] \simeq H^2(B\mathbb{T}, k)$.*

Proof. Let K_{**} be the bicomplex (2.9), with the obvious augmentation $K_{**} \to \underline{k}$. For M a cyclic object in an additive category, we define the operator $b'\colon M_n \to M_{n-1}$ by

$$b' = b - (-1)^n d_n = \sum_{i=0}^{n-1} (-1)^i d_i.$$

Let L_{**} be the $(2,0)$-periodic first-quadrant bicomplex

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
k[\Lambda_2] \arrow[d, "b"'] & k[\Lambda_2] \arrow[l, "\mathrm{id}-t"'] \arrow[d, "-b'"'] & k[\Lambda_2] \arrow[l, "N"'] \arrow[d, "b"'] & \cdots \arrow[l] \\
k[\Lambda_1] \arrow[d, "b"'] & k[\Lambda_1] \arrow[l, "\mathrm{id}-t"'] \arrow[d, "-b'"'] & k[\Lambda_1] \arrow[l, "N"'] \arrow[d, "b"'] & \cdots \arrow[l] \\
k[\Lambda_0] & k[\Lambda_0] \arrow[l, "\mathrm{id}-t"'] & k[\Lambda_0] \arrow[l, "N"'] & \cdots \arrow[l]
\end{tikzcd}$$

with the obvious augmentation $L_{**} \to \underline{k}$, and let M_{**} be the bicomplex obtained from L_{**} by annihilating the even-numbered columns. Let $\phi\colon \operatorname{Tot} K_{**} \to \operatorname{Tot} L_{**}$ be the map induced by $(\mathrm{id}, s_{-1}N)\colon k[\Lambda_n] \to k[\Lambda_n] \oplus k[\Lambda_{n+1}]$, and let $\psi\colon \operatorname{Tot} L_{**} \to \operatorname{Tot} M_{**}$ be the map induced by $-s_{-1}N + \mathrm{id}\colon k[\Lambda_n] \oplus k[\Lambda_{n+1}] \to k[\Lambda_{n+1}]$. A straightforward computation shows that ϕ and ψ are chain maps and that we have a commutative diagram with exact rows

(2.11)
$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Tot} K_{**} \arrow[r, "\phi"] \arrow[d] & \operatorname{Tot} L_{**} \arrow[r, "\psi"] \arrow[d] & \operatorname{Tot} M_{**} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \underline{k} \arrow[r, "\mathrm{id}"'] & \underline{k} \arrow[r] & 0 \arrow[r] & 0.
\end{tikzcd}$$

From the identity $s_{-1}b' + b's_{-1} = \mathrm{id}$, we deduce that each column of M_{**} has zero homology, and hence that that $\operatorname{Tot} M_{**} \simeq 0$. Next we show that each row of L_{**} has zero positive homology, so that the homology of $\operatorname{Tot} L_{**}$ can be computed as the homology of the zeroth column of horizontal homology of L_{**}. This can be proved pointwise, so consider a part of the nth row evaluated at $[m]$:

$$\cdots \to k[\Lambda(n,m)] \xrightarrow{\mathrm{id}-t} k[\Lambda(n,m)] \xrightarrow{N} k[\Lambda(n,m)] \to \cdots . \tag{2.12}$$

By the structure theorem for Λ, we have $\Lambda(n,m) = C_{n+1} \times \Delta(n,m)$, where C_{n+1} is the set of automorphisms of $[n]$ in Λ. Thus, (2.12) is obtained from the complex

$$\cdots \to k[C_{n+1}] \xrightarrow{\mathrm{id}-t} k[C_{n+1}] \xrightarrow{N} k[C_{n+1}] \to \cdots . \tag{2.13}$$

by tensoring with the free k-module $k[\Delta(n,m)]$, and we need only prove that (2.13) is exact. Let

$$x = \sum_{i=0}^{n} x_i c^i \in k[C_{n+1}].$$

Suppose first that $x(\mathrm{id}-t) = 0$; then $x_i = (-1)^{ni}x_0$ and hence $x = x_0N$. Suppose next that $xN = 0$, i.e., that $\sum_{i=0}^{n}(-1)^{ni}x_{n-i} = 0$; putting $y_0 = x_0$ and $y_i = x_i + (-1)^n y_{i-1}$ for $i > 0$, we find $x = y(\mathrm{id}-t)$. This proves the exactness of (2.13), and also that the image of $\mathrm{id}-t\colon k[C_{n+1}] \to k[C_{n+1}]$ is exactly the kernel of the surjective map $k[C_{n+1}] \to k$, $x \mapsto \sum_{i=0}^{n}(-1)^{ni}x_{n-i}$. This map identifies the 0th homology of the nth row of L_{**} evaluated at $[m]$ with $k[\Delta(n,m)]$. Moreover, the vertical map $k[\Delta(n,m)] \to k[\Delta(n-1,m)]$ induced by $-b$ is the usual differential associated with the simplicial k-module $k[\Delta^m]$. This proves that $\operatorname{Tot} L_{**} \to \underline{k}$ is a resolution of k. From (2.11) we deduce that $\operatorname{Tot} K_{**} \to \underline{k}$ is a quasi-isomorphism.

To prove the second statement, we contemplate the complex $\operatorname{Hom}(\operatorname{Tot} K_{**}, \underline{k})$: it is the total complex of the bicomplex

$$\begin{tikzcd}
 & & k \arrow[d, "0"'] \\
 & k \arrow[d, "0"'] & k \arrow[l] \arrow[d, "\mathrm{id}"'] \\
k \arrow[d, "0"'] & k \arrow[l] \arrow[d, "\mathrm{id}"'] & k \arrow[l] \arrow[d, "0"'] \\
\vdots & \vdots & \vdots
\end{tikzcd}$$

with trivial horizontal differentials and alternating vertical differentials. We immediately check that

$$\operatorname{Tot} K_{**} \xrightarrow{\delta} (\operatorname{Tot} K_{**})[2] \to \underline{k}[2]$$

is a cocycle generating the second cohomology module. □

It follows from Lemma 2.10 that the right-hand side of (2.7) is the constant cocyclic k-module $\underline{k}$ with $k[u]$-comodule structure classified by $\delta\colon \underline{k} \to \underline{k}[2]$. Comparing with (2.8) and noting that δ is natural in k, we deduce that Theorem 2.4 holds by choosing $\gamma \in H_1(\mathbb{T}, \mathbb{Z})$ to be the generator corresponding to $\delta \in H^2(B\mathbb{T}, \mathbb{Z})$.

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